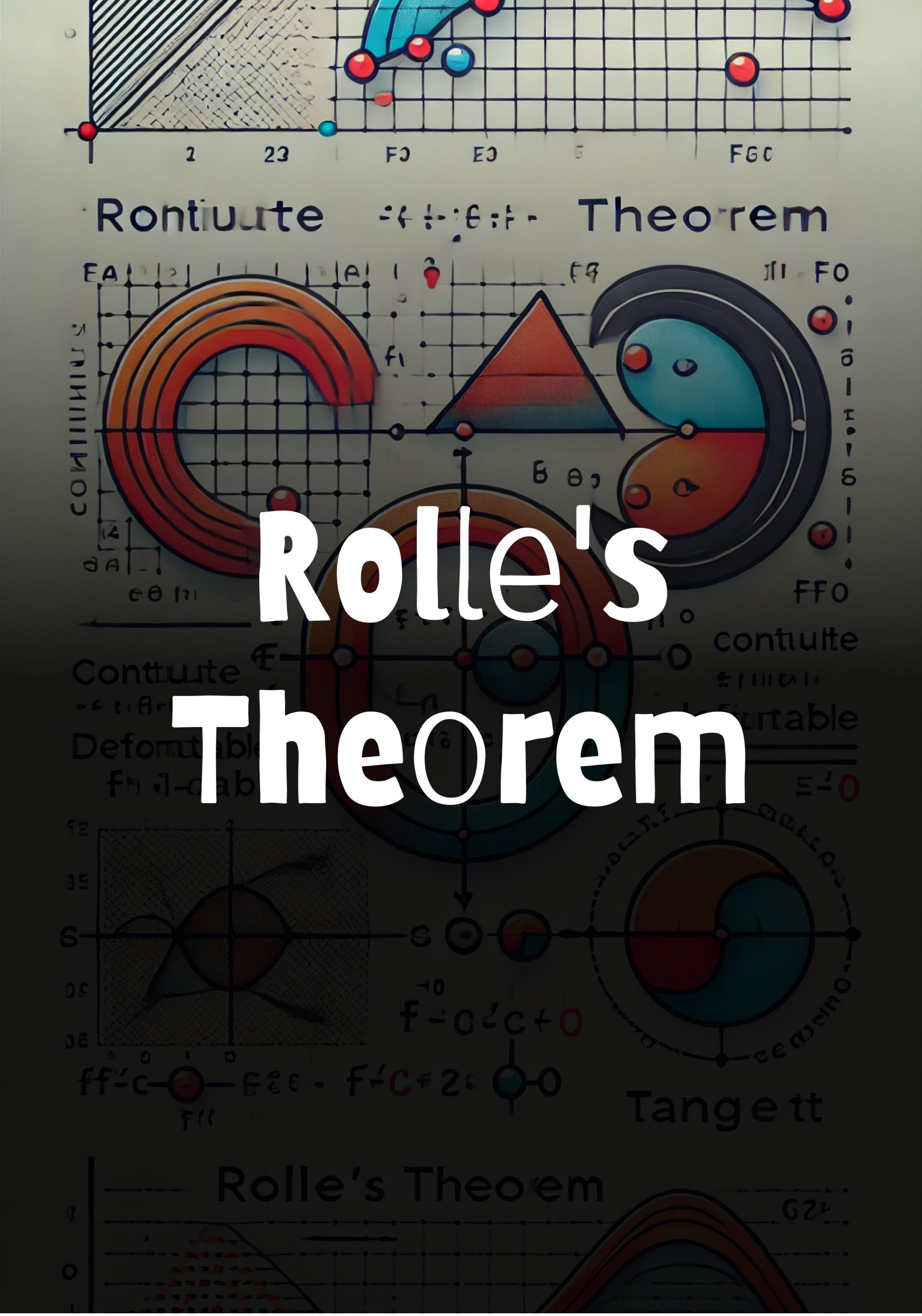


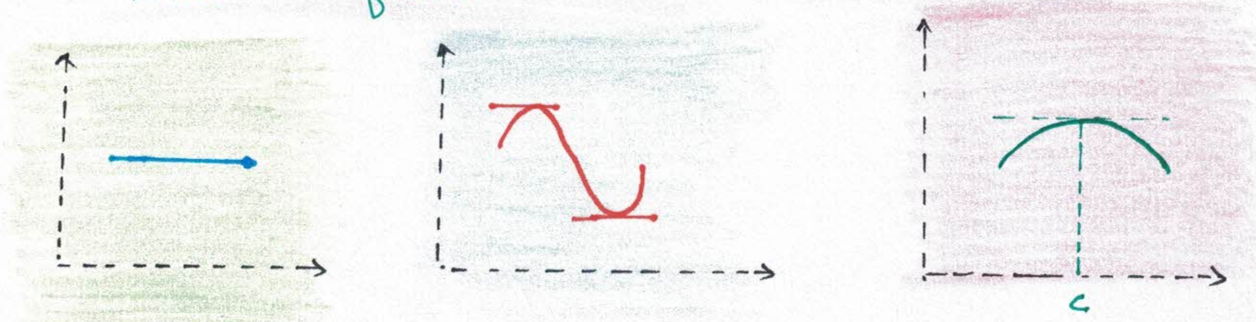
Rolle's Theorem

ROLLE'S THEOREM

Let $f(x)$ be a function defined on $[a, b]$ such that –

1. $f(x)$ is continuous in $[a, b]$
2. $f(x)$ is diff^n in (a, b)
3. $f(a) = f(b)$

then, there exist at least one $c \in (a, b)$ such that $f'(c) = 0$



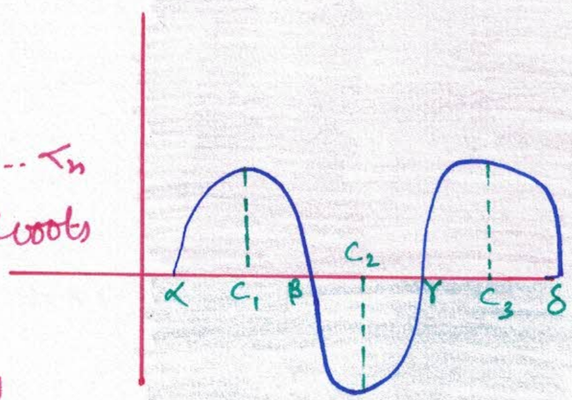
ALGEBRAIC MEANING OF ROLLE'S THEOREM

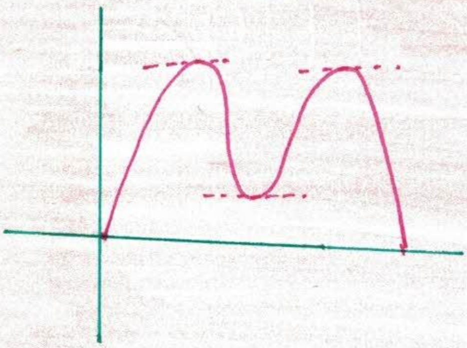
$f(\alpha) = f(\delta) = 0$ in $x \in [\alpha, \delta)$

$f'(c) = 0$ in $c \in [\alpha, \delta]$

If $f(x) = 0$ has n roots $\alpha_1, \alpha_2 \ldots \alpha_n$ then $f'(x) = 0$ has at least $(n-1)$ roots which lies b/w roots of $f(x) = 0$

i.e. $f'(x) = 0$ i.e $c_1, c_2 \ldots c_{n-1}$





$f(x)=0$ has two roots

$f'(x)=0$ has three roots

c_1 lies b/w α_1 & α_2

c_2 lies b/w α_2 & α_3

⋮

c_{n-1} lies b/w α_{n-1} & α_n

Question

Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$ Also find 'c' of Rolle's theorem.

$$f(-3) = -3 \times 0\, e^{-3/2} = 0$$

$$f(0) = 0$$

Cont^n & Diff^n $\longrightarrow$ Rolle's theorem applicable

$$f(x) = (x^2+3x)e^{-x/2}$$

$$f'(x) = (x^2+3x) \times \frac{-1}{2} \times e^{\frac{-x}{2}} + e^{-x/2}(2x+3) = 0$$

$$+\frac{(x^2+3x)e^{-x/2}}{2} = +e^{-x/2}(2x+3)\quad e^{-x/2} \neq 0$$

$$x^2+3x = 4x+6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, -2 \qquad x \neq 3$$

$$\boxed{x = -2}$$

Question

If $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots\cdots \frac{a_n}{n+1} = 0$, then P.T. at least one root of $a_0 + a_1x + \cdots\cdots a_nx^n = 0$ lies b/w $(0,1)$.

$$f'(x) = a_0 + a_1x + a_2x^2 + \cdots\cdots a_nx^n$$

$$f(x) = a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \cdots\cdots \frac{a_nx^{n+1}}{n+1} + K$$

All Polynomials are Continuous & diff^n

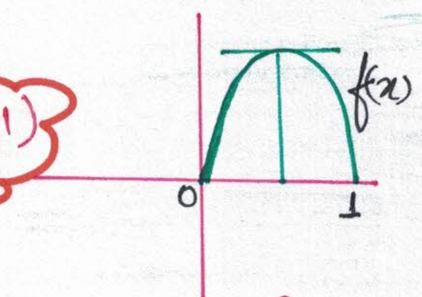
$$f(0) = K$$

$$f(1) = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots\cdots \frac{a_n}{n+1} = 0 + K = K$$

given

Rolle's Theorem applicable

$\therefore$ $f'(x) = 0$ has one root b/w $(0,1)$



Question

If $a_1 + a_2 + a_3 + a_4 = 0$ then P.T. $a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6 = 0$ has at least one root b/w $(-1,+1)$ for any $b_1, b_2, b_3 \in R$.

$\rightarrow$ Let $f'(x) = a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6$

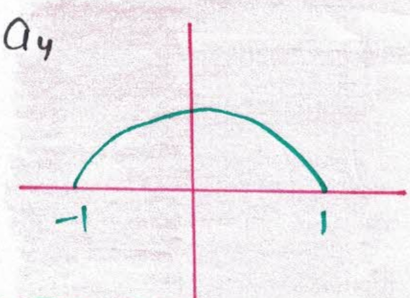
$$f(x) = a_1x + \frac{b_1x^2}{2} + a_2x^3 + \frac{b_2x^4}{4} + a_3x^5 + \frac{b_3x^6}{6} + a_4x^7$$

$$f(-1) = -a_1 + \frac{b_1}{2} + \frac{b_2}{4} - a_2 - a_3 + \frac{b_3}{6} - a_4$$

$$= \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}$$

$$f(1) = a_1 + \frac{b_1}{2} + a_2 + \frac{b_2}{4} + a_3 + \frac{b_3}{6} + a_4$$

$$f(1) = \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{6}$$

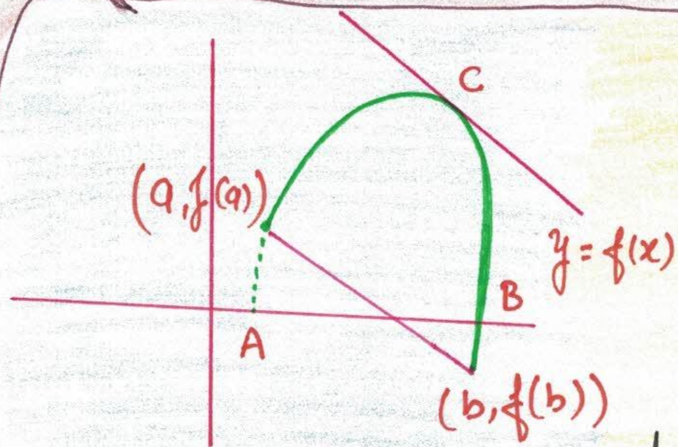


One root b/w $(-1,1)$

LANGRANGE'S MEAN VALUE THEOREM

NOTES

NOTES



$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

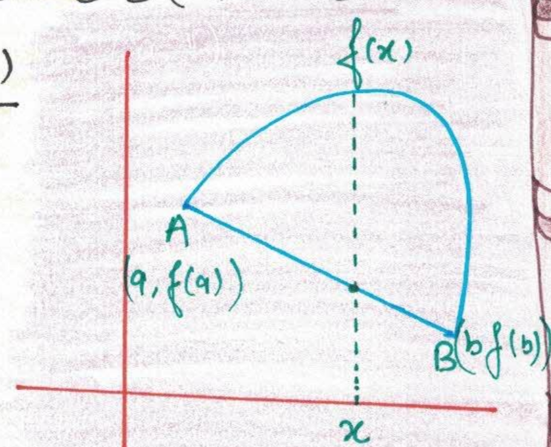
for $c \in (a,b)$

Let $f(x)$ be a function defined on $[a,b]$ such that –

- $f(x)$ is continuous in $[a,b]$
- $f(x)$ is diff^n in (a,b)

Then there exists at least one $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Eq of AB →

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$H(x) = f(x) - y$$

$$H(x) = \left[f(x) - f(a)\right] - \left[\frac{f(b)-f(a)}{b-a}(x-a)\right]$$

$H(x) \rightarrow$ continuous & diff^n

$H(a) = 0$

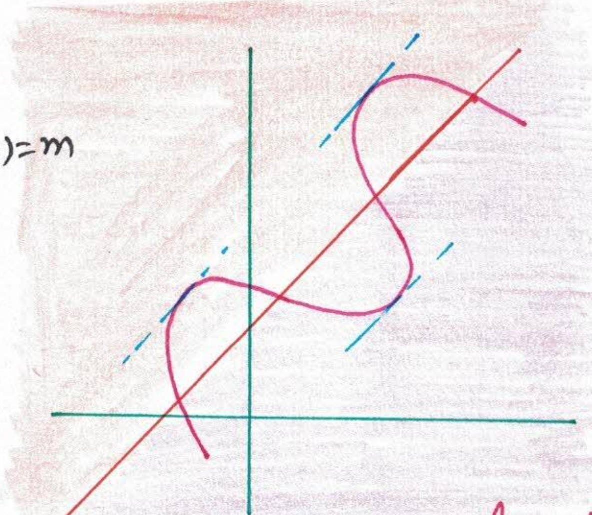
$H(b) = 0$

$H(x)$ satisfies all condition of Rolle's Theorem in $[a,b]$
So $H'(x)$ has at least one root in (a,b) such that
$H'(c)=0$

Question

If the curve $y=f(x)$ intersect the line $y=mx+c$ at 'n' distinct points then find min. possible roots of $f''(x)=0$

$\longrightarrow$ Let us consider P.O.I are 4.



$f'(x)=m$ has 3 roots

So, if P.O.I are n, then $f'(x)=m$ has n-1 roots.

$\boxed{f''(x) \text{ has } n-2 \text{ roots}}$

Question

Prove that the eq^n $x\cos x=\sin x$ has at least one root in $(\pi, 2\pi)$

$\longrightarrow$ Let $f(x)=\frac{\sin x}{x}$ $\rightarrow$ Cont^n & diff^n in $(\pi, 2\pi)$

$f'(\pi) = f(2\pi)=0$

$f'(x) = \frac{x\cos x-\sin x}{x^2}=0$

$$\boxed{x\cos x=\sin x}$$

Question Prove that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b)-f(a)}{b^3-a^3}$ for some $\in (a,b)$ if $f(x)$ is cont. in $[a,b]$ & diff^n in (a,b).

Let $g(x) = f(x) - f(a) + A(x^3 - a^3)$

$g(a) = 0$ then $g(b)$ must be zero

$g(b) = g(b) - f(a) + A(b^3 - a^3) = 0$

$$A = \frac{f(a)-f(b)}{b^3-a^3}$$

$g'(c) = 0$

$g'(c) = f'(c) + A(3c^2 - 0) = 0$

$f'(c) = -3Ac^2$

$$\boxed{\frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3-a^3}}$$

Question Let $f(x)$ be a cont^n & diff^n $\forall x$ and $f(1) = 0$ then P.T. $\exists\, c \in (0,1)$ such that $cf'(c) + f(c) = 0$

Let $f(x) = x f(x)$

$f(0) = 0 \quad f(1) = f(1) \times 1 = 0$

$f'(x) = x f'(x) + f(x) = 0$

$$\boxed{f'(x) = 0 \ \exists\ c \in (0,1)}$$

Question

Let $f(x)$ be a poly whose two consecutive roots are a & b $(a<b)$ then P.T $\exists$ $c \in (a,b)$ such that $f'(c) + 2014 f(c) = 0$.

$$f(a) = f(b) = 0$$

$$g(x) = e^{2014x} f(x)$$

$$g'(x) = e^{2014} f'(x) + f(x) \times 2014 e^{2014x}$$

$$g'(x) = e^{2014} f'(c) + f(c) \times 2014 e^{2014c} = 0$$

$$\boxed{g'(c) + f(c) \times 2014 = 0}$$

Question

If f & g be two cont^n & diff^n $\forall$ x and $g(x) \neq 0$ in (a,b), $g'(x) \neq 0$ in (a,b) then P.T -

$$\frac{g(a) f(b) - g(b) f(a)}{g(c) f'(c) - g'(c) f(c)} = \frac{(b-a)[g(a)\, g(b)]}{(g(c))^2}$$

for some $c \in (a,b)$.

$$\rightarrow \frac{g(c) f'(c) - g'(c) f(c)}{(g(c))^2} = \frac{g(a) f(b) - g(b) f(a)}{(b-a)\, g(a)\, g(b)}$$

$$= \left[\frac{f(b)}{g(b)} - \frac{f(b)}{g(a)}\right] \times \frac{1}{b-a}$$

Let $h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2} = \boxed{\frac{g(c) f'(c) - f(c) g'(c)}{(g(c))^2}}$$

INEQUALITIES RELATED TO LMVT/ROLLE'S THEOREM

Let $f(x)$ be continuous & diffⁿ function in $[a, b]$

and $m \leq f'(x) \leq M \ \forall x \in (a, b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$m < \frac{f(b)-f(a)}{b-a} \leq M$$

$$m(b-a) \leqslant f(b) - f(a) \leqslant M(b-a)$$

$$\boxed{f(a) + m(b-a) \leqslant f(b) \leqslant M(b-a) + f(a)}$$

Question If $f(0) = -2$ and $f'(x) \geqslant 5 \ \forall x$ then find the max/min value of $f(2)$.

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

For $[0, 2]$, $\dfrac{f(b)-f(a)}{b-a} \geqslant 5$

$$\frac{f(2) - f(0)}{2-0} \geqslant 5$$

$$\frac{f(2)+2}{2} \geqslant 5 \Rightarrow f(2)+2 \geqslant 10$$

$$f(2) \geqslant 8$$

$$\boxed{f(2)\Big|_{min} = 8}$$

Question If $f(0)=2$ and $f'(x) \leq 3 \ \forall x$ then find max | min of $f(2)$.

$\longrightarrow$ $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

In $[0,2]$, $f'(x) = \dfrac{f(2)-f(0)}{2-0} \leq 3$

$f(2) - 2 \leq 6$

$f(2) \leq 8$

$$\boxed{f(2)|_{max} = 8}$$

Question If $f(0)=0$ and $f'(x) = \dfrac{1}{1+x^2} \ \forall x$ P.T. $\dfrac{2}{5} < f(2) < 2$.

$\longrightarrow$ $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1+[0,4]} = \dfrac{1}{[1,5]} = \left[\dfrac{1}{5}, 1\right]$

$x \in (0,2)$

$f'(x) = \dfrac{f(b)-f(a)}{b-a}$

$\dfrac{f(2)-0}{2-0} \in \left(\dfrac{1}{5}, 1\right)$

$$\boxed{f(2) \in \left(\dfrac{2}{5}, 2\right)}$$

Let $f(x)$ be cont^n & diff^n in $[a,b]$ such that

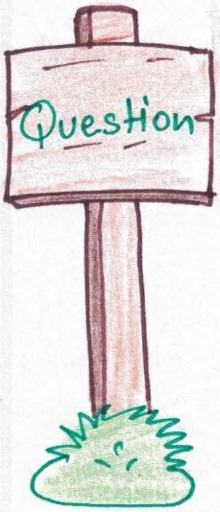
$|f'(x)| \leq M \ \forall x \in [a,b]$

$\left|\dfrac{f(b)-f(a)}{b-a}\right| \leq M$

$$\boxed{|f(b)-f(a)| \leq M\,|b-a|}$$

If $|f'(x)| \geq m$

$$\boxed{|f(b)-f(a)| \geq m(b-a)}$$



For $x, y \in R$ & $x \neq y$ P.T. $|\sin x - \sin y| \leq |x-y|$

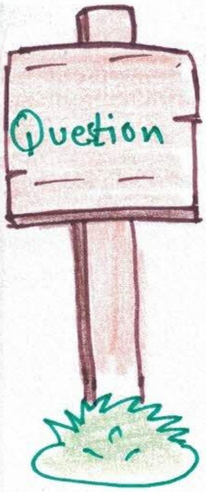
$\rightarrow$

$$f'(x) = \sin x$$

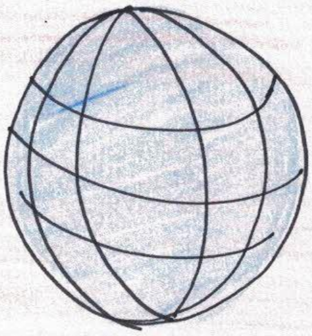
$$|f'(x)| = |\sin x| \leq 1$$

$$\text{for } (x, y) \Rightarrow \frac{|\sin x - \sin y|}{|x-y|} \leq 1$$

$$\boxed{|\sin x - \sin y| \leq |x-y|}$$



P.T $(\beta-\alpha)\sec^2\alpha < (\tan\beta - \tan\alpha) < (\beta-\alpha)\sec^2\beta$.
for $\left(0 < \alpha < \beta < \frac{\pi}{2}\right)$



$$f(x) = \tan(x) \quad x \in (\alpha, \beta)$$

$$f'(x) = \sec^2 x$$

$$\sec^2 x = \frac{\tan\beta - \tan\alpha}{\beta-\alpha}$$

$$f''(x) = 2\sec^2 x \tan x > 0 \quad \left[x \in \left(0, \frac{\pi}{2}\right)\right]$$

$f'(x)$ is $\uparrow$ function

$$\alpha < x < \beta$$

$$f'(\alpha) < f'(x) < f'(\beta)$$

$$\sec^2\alpha < \frac{\tan\beta - \tan\alpha}{\beta-\alpha} < \sec^2\beta$$

$$\boxed{(\beta-\alpha)\sec^2\alpha < \tan\beta - \tan\alpha < \sec^2\beta\,(\beta-\alpha)}$$

Question

If $f(x) = \frac{1}{x}$, then find the value of b for which LMVT is true in $[a, b]$ where $a = 1$

$$f'(x) = \frac{-1}{x^2}$$

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{-1}{x^2} = \frac{f(b)-1}{b-1}$$

$$\frac{-1}{x^2} = \frac{\frac{1}{b}-1}{b-1} = \frac{1-b}{b(b-1)} = \frac{-1}{b}$$

$$\frac{-1}{x^2} = \frac{-1}{b} \Rightarrow x^2 = b$$

$$c^2 = b \Rightarrow c = \pm\sqrt{b}$$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$1 < c^2 < b^2$$

$$b^2 > c^2$$

$$b^2 > b$$

$b(b-1) > 0$

$+ \quad | \quad - \quad | \quad +$
$ 0 1$

$b > 1$ $\boxed{b \in (1, \infty)}$

Question

If $f(x)$ is cont^n and diff^n in $[0,2]$ & $f(0) = 2$, $f(2) = 8$ and $f'(x) \leq 3 \ \forall \ x \in [0,2]$ find $f(1)$.

$$f'(x) = \frac{f(2) - f(1)}{1}$$

$$f'(x) = 8 - f(1)$$

$$f'(x) = \frac{f(1) - f(0)}{1}$$

$$f'(x) = f(1) - 2$$

$8 - f(1) \leq 3$ $\qquad$ $f(1) - 2 \leq 3$

$f(1) \geq 5$ $\qquad$ $f(1) \leq 5$

$$\boxed{f(1) = 5}$$